Research Statement

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Introduction

Geometric Group Theory, my principal area of research, explores the interplay between finitely generated groups and the geometry of spaces on which they act. Arthur Cayley and Max Dehn were some of the first to develop ideas in Geometric Group Theory. More recently, Mikhail Gromov revolutionized the field with a wealth of important contributions including his polynomial growth theorem and the introduction of δ–hyperbolic groups.

I am specifically interested in the coarse geometry of groups. The natural notion for a coarse equivalence between metric spaces is that of quasi-isometry. Two spaces are quasi-isometric if one can be mapped onto the other by a function differing from an isometry by an additive and multiplicative constant. Much of a group's algebra and geometry can be recovered from only the quasi-isometry class of its Cayley graph. With this in mind, Gromov proposed a program to study groups up to quasi-isometry.

An important quasi-isometry invariant of a metric space is its *divergence* function. Given a metric space X and $r > 0$, the divergence function $Div(X, r) = Div(X)$ is the supremum over all lengths of minimal paths, which avoid a ball of radius r, connecting two points that are distance roughly r apart. One can think of the divergence function as the best upper bound on the rate a pair of geodesic rays can stray apart from one another. For a finitely generated group G, $Div(G)$ is the divergence of its Cayley graph with the word metric.

Groups which are δ–hyperbolic, an important class of groups possessing properties of negative curvature, all exhibit at least exponential divergence. On the other end of the spectrum, $\mathbb{Z}^n$ displays linear divergence for $n \geq 2$. In some sense, the divergence function measures the presence of negative curvature in a given group.

Gromov conjectured that groups with non-positive curvature, such as CAT(0) groups, should exhibit either linear or exponential divergence [Gro93a]. This turns out not to be the case. Many important class of groups such as 3–manifold groups, the mapping class group of a closed surface of genus $g \geq 2$, right-angled Artin groups and right-angled Coxeter groups have been shown to exhibit quadratic divergence [Ger94] [KL98] [Beh06] [BC12] [DT15]. More recently, there have been examples of CAT(0) groups exhibiting polynomial divergence of degree d, for any integer d [BH16] [BD14] [DT15] [Mac13]. Additionally, there are even constructions of groups with divergence function strictly between linear and quadratic [OOS09].

Thick metric spaces of order d are an important class of spaces which can be constructed through a d–step inductive gluing procedure, with initial pieces of linear divergence. An important consequence is that these spaces must have divergence bound above by a polynomial of degree $d+1$ [BD14].

Lower bounds on divergence in general is a difficult problem, and there are not many general results in this direction. Many of the specific examples in the literature are treated by a case-by-case analysis to provide such a lower bound in that setting, and it often requires significant technical work. Much of my research is geared to develop general tools to understand divergence. One consequence of which is I provide geometric and group-theoretic criteria which imply exact bounds on divergence.

I have mostly focused on exploring divergence questions in *right-angled Coxeter groups* and *CAT(0) cube complexes*. Both these topics have played a central role in contemporary mathematics. I provide a section dedicated to each giving a short background, some of my results, and directions for future research.

Right-Angled Coxeter Groups

Associated to any simplicial graph Γ is a *right-angled Coxeter group* (RACG for short), W_Γ, whose presentation consists of an order 2 generator for each vertex of Γ with the relation that two generators commute if there is an edge between the corresponding vertices of Γ.

Despite their simple presentation, RACGs form a wide class of groups. Free groups, free abelian groups and many surface groups are finite index subgroups of a RACG. Furthermore, every right-angled Artin group is finite index in some RACG [DJ00]. Incredibly, Agol and Wise show that the fundamental group of every hyperbolic 3–manifold is virtually a subgroup of a RACG. A fundamental question which motivates my research is the following:

Question 2.1. *What are the quasi-isometry classes of right-angled Coxeter groups?*

The answer to the above question is necessarily very complex and not much is currently known even in simplified settings. However, understanding the divergence function, a quasi-isometry invariant, of these groups allows us to make progress.

Dani-Thomas studied divergence in RACGs for the case when Γ does not contain triangles. These RACGs are precisely those for which the Davis complex, a natural CAT(0) space a RACG acts on, is 2–dimensional. These authors prove that in the 2–dimensional case, RACGs exhibiting quadratic divergence are exactly those whose graph is CFS and does not decompose as a join [DT15]. The CFS condition, "constructed from squares", is a purely graph theoretic criteria which can be computationally checked.

A classification still remained open for RACGs of higher dimension. A resolution to this question was especially important for the theory of random RACGs as a "randomly selected RACG" has dimension larger than 2 with probability 1. I prove such a classification extends to all RACGs:

Theorem 2.2 (Levcovitz; [Lev])**.** *Suppose Γ is not a nontrivial join. $Div(W_\Gamma)$ is quadratic if and only if Γ is CFS. Additionally, if Γ is not CFS, then $Div(W_\Gamma)$ is at least cubic.*

The authors of [BFRHS] show $Div(W_\Gamma)$ is linear if and only if Γ is a nontrivial join. Hence, the above theorem exactly classifies quadratic divergence in RACGs. A priori, a RACG could exhibit divergence strictly between a quadratic and cubic polynomial. However, the above theorem proves this cannot be the case.

A random graph, $\Gamma(n, p(n))$, in the Erdős-Rényi model is a n vertex graph having an edge between a given pair of vertices with probability $p(n)$. A random RACG is simply the RACG associated to a random graph. Behrstock–Falgas-Ravry–Hagen–Susse give a threshold theorem for when a random graph is CFS with probability 1. Combining their result with the classification of quadratic divergence, given by the theorem stated above, we obtain a threshold function for the transition between quadratic to at least cubic divergence in random RACGs.

Theorem 2.3 (Behrstock–Falgas-Ravry–Hagen–Susse, Levcovitz; [BFRHS] [Lev])**.** *Suppose $p(n)$ is a probability density function bounded away from 1 and let $\epsilon > 0$. Let $\Gamma = \Gamma(p(n), n)$ be a random graph. If $p(n) > n^{-\frac{1}{2}+\epsilon}$, then the right-angled Coxeter group W_Γ asymptotically almost surely exhibits quadratic divergence. If $p(n) < n^{-\frac{1}{2}-\epsilon}$, then W_Γ asymptotically almost surely exhibits at least cubic divergence.*

Using more general results explained in the next section, for each positive integer, d, I provide a graph-theoretic criteria on Γ which implies the divergence of W_Γ is polynomial of degree d. To obtain a deeper understanding of these groups, I have been working on the following project which seems just in reach:

Question 2.4. *For W_Γ not relatively hyperbolic, is it always the case that $Div(W_\Gamma)$ is a polynomial of positive integer degree? Give a graph-theoretic criteria which determines the divergence of any given RACG.*

Graph-theoretic criteria obtained from the above project will open the door for new results on random groups regarding probability densities for which a random right-angled Coxeter group exhibits a certain divergence function. Additionally, computer software can be developed to detect such criteria in a graph, providing a computational tool to study right-angled Coxeter groups. In fact, other researchers and I have already written such code to study RACGs using the currently known results.

My work on RACGs can be extended to other families of groups. Coxeter groups (not necessarily right-angled) are a natural generalization of Euclidean, hyperbolic and spherical reflection groups. Much work is still needed in this general setting as even a classification of linear divergence is unknown. Using these groups' wall structure, I prove preliminary results relating graph-theoretic properties with bounds on divergence. In particular, I prove:

Theorem 2.5 (Levcovitz; [Lev])**.** *Given a positive integer d, there are infinitely many Coxeter groups which are not right angled and exhibit polynomial divergence of degree d.*

CAT(0) Cube Complexes

A cube complex is obtained by gluing a set of unit cubes, of possibly differing dimension, along their faces by isometries. A *CAT(0) cube complex* is a simply connected cube complex satisfying the CAT(0) inequality (a condition guaranteeing desirable properties of nonpositive curvature). Some examples of CAT(0) cube complexes are $\mathbb{R}^n$ tiled by n-cubes and the rich subclass of trees. Every CAT(0) cube complex comes innately equipped with a set of *hyperplanes*, each of which separate the space into two disjoint components.

CAT(0) cube complexes have recently played a central role in mathematics. The resolution of the virtual Haken conjecture, an outstanding conjecture of Thurston, by Agol relied heavily on Wise's CAT(0) cube complex developments. Furthermore, the class of groups which act non-trivially on a CAT(0) cube complex is surprisingly large.

The Rank Rigidity Theorem [CS11], is a pivotal result in the field showing the existence of, in some sense, "negatively curved geodesics" in an essential, locally-finite, irreducible CAT(0) cube complex with cocompact automorphism group. As a consequence, the divergence of these spaces is either linear or at least quadratic [Hag13]. For these cube complexes, the following broad question still remains:

Question 3.1. *Given an essential locally-finite CAT(0) cube complex X with cocompact automorphism group, what possible values can $Div(X)$ take?*

In my work, I define two different notions for when a pair of hyperplanes are sufficiently separated; namely, these are *degree d separated hyperplanes* and *chain-separated hyperplanes*. I show that the existence of just a pair of such hyperplanes in a CAT(0) cube complex implies lower bounds, which are often sharp, on the divergence of the cube complex. For instance, using the first of these notions I prove:

Theorem 3.2 (Levcovitz; [Lev])**.** *Suppose X is an essential, locally compact CAT(0) cube complex with cocompact automorphism group containing a pair of degree d separated hyperplanes, then $Div(X)$ is at least a polynomial of degree $d + 1$.*

Using the above theorem together with upper bounds provided by thickness, I prove there are many RACGs exhibiting polynomial divergence of any positive integer degree. These RACGs, in particular, include all previous examples where divergence results were known.

The existence of chain-separated hyperplanes provides a different bound on divergence:

Theorem 3.3 (Levcovitz; [Lev])**.** *Suppose X is an essential, locally compact CAT(0) cube complex with cocompact automorphism group containing a pair of chain-separated hyperplanes, then $Div(X)$ is at least $log(log(r))r^2$.*

The above two theorems reduce the problem of bounding the divergence of a CAT(0) cube complex to one of finding a pair of hyperplanes with certain separation properties.

As an approach to Question 3.1, I am investigating when chain-separated hyperplanes can occur in general. I believe this approach has a chance of showing an additional gap in the spectrum of possible divergence functions for these cube complexes.

Further Work

There are many different quasi-isometry classes of RACGs exhibiting the same divergence function; hence, the divergence function alone is not sufficient to coarsely distinguish RACGs (Question 2.1). A more refined approach is to distinguish the divergence for different directions in the Cayley graph of a RACG. The contracting boundary, also a quasi-isometry invariant, encodes directions of super-linear divergence in a CAT(0) group [CS15]. I am working on the following question which aims to provide a variation of the contracting boundary that is more sensitive to different divergence functions.

Question 4.1. *Develop a quasi-isometry invariant boundary, or series of boundaries, which distinguish directions of different polynomial degrees of divergence in a CAT(0) space.*

Behrstock-Hagen-Sisto prove right-angled Coxeter groups are either thick or hyperbolic relative to thick subgroups. An analogous result may hold in the generality of cocompact CAT(0) cube complexes due to their strong rigidity properties, leading to the following project:

Question 4.2. *Are cocompact CAT(0) cube complexes either thick or hyperbolic relative to thick subcomplexes?*

There are very recent cube complex results developed by Hagen-Susse [HS16] which show cocompact CAT(0) cube complexes come equipped with a factor system. I am currently researching how product regions in factor systems are pieced together. These product regions should form the base spaces used in an inductive construction of a thick structure for these cube complexes.

Artin groups are a natural relative to Coxeter groups. These groups have the same presentations as Coxeter groups except no torsion condition is placed on the generators. As natural as this extension is, not much is known about these groups. For instance, the following is surprisingly still open:

Question 4.3. *Do Artin groups have solvable word problem? Are they torsion-free?*

It is known that non-relatively hyperbolic Artin groups exhibit at most quadratic divergence [BD14] [CP14]. The following project requires techniques that are also used to study particular cases of the above broad question.

Question 4.4. *Is there an Artin group exhibiting divergence strictly between linear and quadratic?*

A positive answer would give a rare "natural" finitely presented group with this property. On the other hand, proving such a group cannot exist would likely come from the development of new tools in the study of mysterious Artin groups.

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